1D Brownian motion.

Def 1D Brownian motion:

$\{B(t): 0 \le t < \infty\}$ on $(\Omega, \mathcal{F}, P)$ ($\mathcal{F}$ - σ-algebra, P - probability measure)

$B(t,\omega): 0 \le t < \infty$, $\omega \in \Omega$ satisfies:

a) $B(0,\omega) = 0 \quad \forall \omega$

b) $B(t,\omega)$ - continuous in t $\forall \omega$.

c) $0 < t_1 < t_2 < \dots < t_n$: $B(t_1), B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1})$ - independent, normal, mean zero variance: $t_1, t_2 - t_1, \dots, t_n - t_{n-1}$ - correspondingly.

(c) is equivalent to requiring $E(B(t)) = 0 \;\forall t$, $E(B(t)B(s)) = \min(t,s)$)

Another point of view: Probability measure on $C([0,+\infty))$ such that $B(0) = 0$ a.s. and c) is satisfied.

Norbert Wiener(1894-1964)

Theorem (Wiener) Brownian motion exists.

Properties:

1) (Uniqueness) Any two equivalent up to measure-preserving transformation between Ω and Ω_1.

2) (Markov Property) $\left(B_{t+s} - B_s\right)_{t>0}$ – Brownian motion.

3) $-B_t$ – Brownian motion

4) (Brownian Scaling): $\forall c>0$: cB_{t/c^2} – Brownian motion.

5) (Time-reversal): Let $X_0 = 0$, $X_t = t\, B_{1/t}$. Then X_t – BM.

<u>Proof</u> Check covariance for $t>0$. $E(t\, B_{1/t} \cdot s\, B_{1/s}) = st\, E(B_{1/t} B_{1/s}) = \min(s,t)$,

Continuity $t>0$ – obvious.

For rational t, X_t distributed like B_t. So $\lim\limits_{\substack{t\to 0 \\ t\in\mathbb{Q}}} X_t = \lim\limits_{\substack{t\to 0 \\ t\in\mathbb{Q}}} B_t = 0$.

But since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\lim\limits_{t\to 0} X_t = 0 = X_0$. ∎

5') $P\left(\lim\limits_{t\to\infty} \frac{B(t)}{t} = 0\right) = 1$.

<u>Proof</u>. $\frac{B(t)}{t} = X\left(\frac{1}{t}\right)$ ∎

Def Quadratic variation

Let (X_t) - real-valued process

X has a finite quadratic variation at time t if

for $\Delta_n = \{0 = t_0 < t_1 < \dots < t_n = t\}$ - partition of $[0,t]$.

$|\Delta_n| = \max(t_i - t_{i-1}) \to 0$. $\exists \; \underset{|\Delta_n| \to 0}{P-\lim} \sum (X_{t_i} - X_{t_{i-1}})^2 =: \langle X, X \rangle_t$

notation.

$\langle X, X \rangle_t$ is an increasing (random) function.

Theorem $\langle B, B \rangle_t = t$.

We will use:

Gian Carlo Wick (1909-1992)

Wick's formula. Let $X_1, \dots, X_n$ be centered normal variables.

Then $E(X_1 \dots X_n) = \sum\limits_{\substack{\text{perfect pairings} \\ \text{of } \{1,\dots,n\}}} \prod E(X_{i_k} X_{j_k})$.

In particular: $E(X^4) = 3(E(X^2))^2$

Proof (of Wick's)

Use Characteristic functions.

If X is centered Normal, then $E(e^{tX}) = e^{\frac{t^2 \sigma}{2}}$, where $\sigma = E(X^2)$.

$E(\prod e^{t_i X_i}) = E(e^{\overbrace{\sum t_i X_i}^{\text{Normal}}}) = \exp\left(\frac{1}{2} E\left(\left(\sum t_i X_i\right)^2\right)\right) = \exp\left(\frac{1}{2} \sum t_i t_j E(X_i X_j)\right)$

On one hand: $E(\prod e^{t_i X_i}) = E(\prod(1 + t_i X_i + t_i^2 X_i^2 + \ \ ) =$

$1 + t_1 t_2 \dots t_n E(\prod X_i) + \dots$

On the other hand:

$\exp\left(\frac{1}{2} \sum t_i t_j E(X_i X_j)\right) = \exp\left(\sum_{i<j} t_i t_j E(X_i X_j) + \frac{1}{2} \sum t_i^2 E(X_i^2)\right) =$

$\prod_{i<j} (1 + t_i t_j E(X_i X_j) + \dots) \prod (1 + \frac{1}{2} t_i^2 E(X_i^2) + \dots)$

let us look for coefficient in front of $t_1 t_2 \dots t_n$. ∎

Proof of Quadratic Variation:

$E\left(\left(\sum (B_{t_i} - B_{t_{i-1}})^2 - t\right)^2\right) = E\left(\left(\sum \left[(B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right]\right)^2\right) =$

L^2-norm!

$E\left(\sum (B_{t_i} - B_{t_{i-1}})^4 - 2 \sum (B_{t_i} - B_{t_{i-1}})^2 (t_i - t_{i-1}) + \sum (t_i - t_{i-1})^2\right) = 3 \sum (t_i - t_{i-1})^2 - 2 \sum (t_i - t_{i-1})^2 + \sum (t_i - t_{i-1})^2 =$

$2 \sum (t_i - t_{i-1})^2 \le 2 |\mathcal{D}_n| t \to 0.$

So $\sum (B_{t_i} - B_{t_{i-1}})^2 \xrightarrow[L^2]{} t \Rightarrow$ converges in Probability ∎

Corollary: A.s. B.M has infinite Variation on any interval $[a,b]$

Proof. Know: $\sum (B_{t_i} - B_{t_{i-1}})^2 \xrightarrow[P]{} b-a$ Select a subsequence so that converges a.s.

Observe: $\sum (B_{t_i} - B_{t_{i-1}})^2 \le \sup_i |B_{t_i} - B_{t_{i-1}}| \cdot \mathrm{Var}_a^b B$

Now pass to the limit: $\sup_i |B_{t_i} - B_{t_{i-1}}| \to 0$ (B - continuous).

So $(b-a) \le 0 \cdot \mathrm{Var}_a^b B$ if $\mathrm{Var}_a^b B < \infty$ — Contradiction! ∎

More is true (no proofs):

1) Thm (Levy). Let $h(t) = (2t\log\frac{1}{t})^{\frac{1}{2}}$.

Then $$P\left(\overline{\lim_{\varepsilon\to 0}} \sup_{\substack{0\le t_1\le t_2\le 1\\ t_2-t_1<\varepsilon}} \frac{|B_{t_2}-B_{t_1}|}{h(\varepsilon)} = 1\right) = 1.$$

2) Law of iterated logarithm:

$$\overline{\lim_{t\to 0}} \frac{B_t}{\sqrt{2t\log\log\frac{1}{t}}} = 1 \quad \text{a.s.}$$

2') $$\lim_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} = 1 \quad \text{a.s (apply 2) to } tB_{1/t}).$$

2'') $$\overline{\lim_{s\to 0}} \frac{B_{t+s}-B_t}{\sqrt{2s\log\log s}} = 1 \quad \text{a.s. for given } t$$ (by Markov).